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THE EMPLOYMENT OF GALERKIN'S VECTOR IN DEFINING  
CONCENTRATED LOADS AND MOMENTS  
IN LINEAR ISOTROPIC ELASTICITY

by

James J. Richardson

April 1972

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*Redstone Arsenal, Alabama*

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III

Ground Equipment and Materials Directorate  
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## **ABSTRACT**

This report contains a discussion of a formulation of Galerkin's vector in isotropic linear elasticity and an employment of this vector in describing concentrated loads and moments. Various forms of these concentrated phenomena are developed by the familiar process of superposition and then by a limit solution technique.

The report is primarily based on work contained in the author's doctoral dissertation written at the University of Illinois at Champaign-Urbana. In addition, methods and results obtained from several unpublished papers by Professor Marvin Stippes of the University of Illinois are incorporated.

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## SYMBOLS

$C_{ik}(D)$	Cauchy's operator
$D_i$	Differential operator $\left(\frac{\partial}{\partial x_i}\right)$
$D^2$	$D_i D_i$ or $\nabla^2$
$\tilde{E}$	Strain dyadic
$F_i$	Body force field
$g_i$	Galerkin's vector
$\tilde{I}$	Idemfactor
$L$	Concentrated load
$S$	State of stress
$\tilde{S}$	Stress dyadic
$T_i$	Surface tractions
$u_i$	Displacement field
$V$	Region of elastic space
$\partial V$	Surface of $V$
$\nabla$	Convention del operator $\left(\frac{\partial}{\partial x_i} i_i\right)$
$\Delta(D)$	Determinant of $C_{ik}(D)$
$\delta_{ij}$	Kronecker delta
$\epsilon_{ijl}$	Permutation symbol
$\mu, \lambda$	Lamé's constants
$\nu$	Poisson's ratio
$\rho$	Mass density

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## 1. Introduction

The equations of equilibrium of an isotropic linear elastic body  $V$  are called Cauchy's equations and may be written as

$$C_{ik}(D) U_k + \rho F_i = 0 \quad (1.1)$$

where  $C_{ik}(D)$  is called Cauchy's operator and is defined below

$$C_{ik}(D) = \mu D^2 \delta_{ik} + (\lambda + \mu) D_i D_k \quad . \quad (1.2)$$

The Cauchy equations are formulated with the assumption that  $V$  is a continuous medium. This ultimately leads to the restriction that  $u_i$  is at least piecewise continuous of class  $C^2$  throughout  $V$ . A solution to Eq. (1.1) which satisfies this restriction is called a regular solution\*. The form of a regular solution to Eq. (1.1) depends upon the shape of the body and the type of loading imposed.

Singular solutions exist, however, which fail to exhibit this continuity at a point, along an arc or over a surface of  $V$ . The purpose of this report is to discuss solutions which are singular in some deleted neighborhood of one or more points in  $V$ . Such a solution has one or more isolated singularities and is called an isolated singular solution.

As an isolated (nonremovable) singular point  $x$  in  $V$  is approached, the limits of  $u_i$  and their derivatives do not exist. In fact, the displacements and stresses at  $x$  are unbounded. The isolated singularity may be considered as the manifestation of some type of load or moment. In this report the physical significance which may be attached to various forms of isolated singularities will be discussed.

It will be readily seen that at least three useful functions are served by furthering one's understanding of isolated singularities of Eq. (1.1). There are numerous cases in which loading is closely approximated by a concentrated force internal to a body. Also, singular solutions may be used as influence functions to obtain regular solutions. This is done, for example, in Betti's [1] adaption of the method of singularities to isotropic elasticity. Finally, the investigation of point defects in metallurgy involves these solutions (e.g., see Simmons et al [2]).

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\*Additionally, if  $V$  is unbounded it is required that the  $u_k = 0(1/r)$  and  $u_{k,j} = 0(1/r^2)$  uniformly in the spherical coordinates  $\cdot, \cdot$  as  $r$  approaches infinity.

First, a solution of Eq. (1.1) must be obtained. Because the Cauchy equations are elliptic (assuming that  $C_{ik}(D)$  is positive definite), techniques analogous to those found in potential theory may be used [3]. In Paragraph 2, Cauchy's equations are written in biharmonic form. This is possible through the employment of the Galerkin vector which is developed in Paragraph 2.

## 2. Galerkin Vector

A very concise derivation of the Galerkin vector was presented by Somigliana [4]. Choosing a transformation  $K_{ij}(D)$  so that

$$u_i = K_{ik}(D) v_k \quad , \quad (2.1)$$

the Cauchy equations may be written as

$$C_{ik}(D) K_{km}(D) v_m + \zeta F_i = 0 \quad . \quad (2.2)$$

Now it is specified that  $K_{km}$  is the matrix of cofactors of Cauchy's vector; that is

$$K_{ip}(D) = \frac{1}{2!} \epsilon_{ijk} \epsilon_{pmn} C_{jm} C_{kn} = \nu D^2 \left[ (\cdot + 2\nu) D_{ip}^2 - (\cdot + \nu) D_i D_p \right] . \quad (2.3)$$

By definition

$$C_{ik}(D) K_{km}(D) = \delta_{im} \cdot (D) \quad (2.4)$$

where  $\cdot(D)$  represents the determinant of  $C_{ik}(D)$

$$\cdot(D) = \nu^2 (\cdot + 2\nu) D^6 \quad . \quad (2.5)$$

Thus, Cauchy's equations may be written in terms of the variable  $v_i$ , which will be called an auxillary vector function

$$\nu^2 (\cdot + 2\nu) D^6 v_i + \zeta F_i = 0 \quad . \quad (2.6)$$

Finally, a new auxilliary vector function is defined as

$$g_i = \nabla^2 v_i \quad (2.7)$$

so that

$$u_i = \mu(\lambda + 2\mu) D^2 g_i - (\lambda + \mu) D_m D_i g_m \quad . \quad (2.8)$$

A particular solution to Eq. (2.6) may be found by using the concept of influence functions which are analogous to Green's functions in potential theory. An insight into this approach can be gained from the one dimensional string problem. Consider a string L units in length which is fixed at both ends and which lies along the  $x_1$  axis. If the string, under a tension  $t$ , is subjected to a transverse unit force applied at  $x_1 = s$ , the equation of equilibrium is

$$tx_{2,11} - \delta(x_1 - s) = 0 \quad . \quad (2.9)$$

$\delta(x_1 - s)$  is the Dirac delta function defined by Chen [5]

$$\delta(x_1 - s) = \begin{cases} 0, & \text{for } x_1 \neq s \\ \infty, & \text{for } x_1 = s \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x_1 - s) dx = 1 \quad .$$

A solution of Eq. (2.9) indicating the vertical displacement of the string is

$$v_2 = K(x_1, s) \quad (2.10)$$

where  $K(x_1, s)$  is called an influence coefficient. Importantly, Eq. (2.10) can be used to solve problems concerning distributed transverse loadings  $q(x_1)$  because it can be shown that the resulting equilibrium equation has the solution

$$v_2 = \int_0^L K(x_1, s) q(s) ds \quad . \quad (2.11)$$

A solution to Eq. (2.6) which corresponds to a body force  $\rho F_i$  can be determined in an identical manner. To avoid complications introduced by finite regions, a body  $V^\infty$  of infinite proportions is chosen. Imagine a body force  $\rho F_i$  which is only nonzero over a finite region  $V$ . Then, if  $S(\underline{x}, \underline{\xi})$  is the solution of

$$\Delta(D) S(\underline{x}, \underline{\xi}) + \delta_3(\underline{x} - \underline{\xi}) = 0 \quad (2.12)$$

where  $\delta_3(\underline{x} - \underline{\xi})$  is the volume Dirac delta function, then a solution to Eq. (2.6) at a point  $\underline{x}$  is

$$q_i(\underline{x}) = \int S(\underline{x}, \underline{\xi}) \rho F_i(\underline{\xi}) dD(\underline{\xi}) \quad . \quad (2.13)$$

Fritz John [6] presents the general form of  $S(\underline{x}, \underline{\xi})$ . The particular form of  $S(\underline{x}, \underline{\xi})$  for Cauchy's equations was found to be [7]

$$S(\underline{x}, \underline{\xi}) = \alpha R(\underline{x}, \underline{\xi}) \quad (2.14)$$

where

$$\alpha = \frac{1}{4\pi\mu^2(\lambda + 2\mu)} , \quad R(\underline{x}, \underline{\xi}) = \left[ (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2 \right]^{\frac{1}{2}} . \quad (2.15)$$

Thus, a particular solution to Eq. (2.6) is

$$q_i = \alpha \int \rho F_i(\underline{\xi}) R(\underline{x}, \underline{\xi}) dD(\underline{\xi}) \quad . \quad (2.16)$$

In this form the Galerkin vector  $q_i$  in Eq. (2.8) yields a solution  $u_i$  which represents the displacement within an infinite linearly elastic body  $V^\infty$  because of a body force field  $\rho F_i$  acting only within a finite region  $V$ .

### 3. Limit Solution

The purpose of this section is to employ the limit solution technique of Sternberg and Eubanks [8] to develop and discuss various types of isolated singularities in linear isotropic elasticity.

Again, entire three dimensional space  $V^\infty$  is considered. This time, imagine a sequence of regions  $\{V^{(n)}\}$  each containing the origin and a corresponding sequence of body forces  $\{\rho F_i^{(n)}\}$  such that  $\rho F_i^{(1)}$  acts over region  $V^{(1)}$ ,  $\rho F_i^{(2)}$  over  $V^{(2)}$ , etc. It is required that each  $\rho F_i^{(n)}$  be of class  $C^2$  in  $V^\infty$  and be zero at any point outside of  $V^{(n)}$ . It has been shown that, under these circumstances, a particular solution  $u_i^{(n)}(\underline{x})$  exists for each  $V^{(n)}$  and corresponding body force field  $\rho F_i^{(n)}$ . In addition, as  $n$  approaches infinity the length of the longest chord within  $V^{(n)}$  must uniformly approach zero.

The transformation represented by Eq. (2.14) is now employed, yielding

$$u_k^{(n)}(\underline{x}) = K_{ki} g_i^{(n)}(\underline{x}) \quad . \quad (3.1)$$

Thus, if  $\underline{z}$  is a typical point within  $V^{(n)}$  and  $\underline{x}$  is any other point within  $V^\infty$ ,

$$g_i^{(n)}(\underline{x}) = \alpha \int R(\underline{x}, \underline{z}) \rho F_i^{(n)}(\underline{z}) dV^{(n)}(\underline{z}) \quad . \quad (3.2)$$

The limiting case of Eq. (3.2) is defined as

$$G_i(\underline{x}) = \lim_{n \rightarrow \infty} g_i^{(n)}(\underline{x}) = \lim_{n \rightarrow \infty} \alpha \int R(\underline{x}, \underline{z}) \rho F_i^{(n)}(\underline{z}) dV^{(n)}(\underline{z}) \quad . \quad (3.3)$$

Richardson [7] showed that, in general,

$$G_{i,jk}(\underline{x}) = \lim_{n \rightarrow \infty} g_{i,jk}^{(n)} \quad . \quad (3.4)$$

Therefore,  $U_k$  computed from  $G_k$  represents the limiting solution, a displacement field with an isolated singularity at the origin. The forms which this solution assumes will be discussed in later sections.

$R(\underline{x}, \underline{z})$  may be expanded in the Taylor's series for three variables shown below

$$R(\underline{x}, \underline{z}) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}}{i!j!k!} D_1^i D_2^j D_3^k (r) \cdot \begin{smallmatrix} i \\ 1 \\ j \\ 2 \\ k \\ 3 \end{smallmatrix}, \quad r = r(\underline{x}) \quad . \quad (3.5)$$

Thus, Eq. (3.3) becomes

$$G(\underline{x}) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}}{i!j!k!} D_1^i D_2^j D_3^k (r) \underset{n \rightarrow \infty}{\text{Lim}} \int \phi F^{(n)}(\underline{z}) \cdot \begin{smallmatrix} i \\ 1 \\ j \\ 2 \\ k \\ 3 \end{smallmatrix} dV^{(n)}(\underline{z}) \quad . \quad (3.6)$$

Note that the summation sign may be removed from under the integral sign because Taylor's series is uniformly convergent. Defining the moment integrals  $\underline{M}^{ijk}$  which are of order  $i+j+k$  and operators  $\cdot^{ijk}$  by

$$\underline{M}^{ijk} = \underset{n \rightarrow \infty}{\text{Lim}} \int \phi F^{(n)}(\underline{z}) \cdot \begin{smallmatrix} i \\ 1 \\ j \\ 2 \\ k \\ 3 \end{smallmatrix} dV^{(n)}(\underline{z}) \quad (3.7)$$

$$\cdot^{ijk} = \frac{(-1)^{i+j+k}}{i!j!k!} D_1^i D_2^j D_3^k \quad , \quad (3.8)$$

where  $i+j+k = p$  which is the order of the operator  $\cdot^{ijk}$ , allows one to write the Galerkin vector as

$$G(\underline{x}) = \sum_{i,j,k=0}^{\infty} \cdot^{ijk}(r) \underline{M}^{ijk} \quad . \quad (3.9)$$

This is a series which represents the Galerkin vector for isolated singularities of all orders. Additionally, by choosing appropriate terms of the series, one can write the Galerkin vector corresponding to specific forms of isolated singularities.

It is of interest to investigate the nature of  $\underline{M}^{ijk}$ . Note that if the sequence of regions  $V^{(n)}$  and the corresponding body force fields  $\rho F_i^{(n)}$  are symmetric about all axes, then the integrals  $\underline{M}^{ijk}$  with an odd  $i$ ,  $j$ , or  $k$  are zero. The shape of the region  $V^{(n)}$  is arbitrary so symmetry can be assumed; however,  $\rho F_i^{(n)}$  are not generally symmetric. Later in this report some manifestations of symmetry of  $\rho F_i^{(n)}$  will be discussed.

In Paragraphs 4, 5, and 6 it will be shown that Eq. (3.9) may be written in the form

$$G_i(\underline{x}) = \phi(r)m_i + \phi_j(r)m_{ij} + \phi_{jk}(r)m_{ijk} + \dots . \quad (3.10)$$

#### 4. Fundamental or First Order Singularity

The term in Eq. (3.10) corresponding to  $i=j=k=0$  is

$$G_i^1(\underline{x}) = o(r) \lim_{n \rightarrow \infty} \int \rho F_i^{(n)}(\underline{z}) dV^{(n)}(\underline{z}) . \quad (4.1)$$

From Eq. (2.8) it is seen that the displacement vector is expressed in terms of the second derivatives of  $g_i$ . Therefore, if  $g_i$  is of  $O(r)$  as  $r \rightarrow 0$  then  $u_i$  will be of  $O(1/r)$  as  $r \rightarrow 0$ . This case shall be called the first order singularity. Superscripts on  $G_i$  will indicate the order of singularity. Defining

$$\phi(r) = o(r), m_i = \lim_{n \rightarrow \infty} \int \rho F_i^{(n)}(\underline{z}) dV^{(n)}(\underline{z}) , \quad (4.2)$$

produces the first term of Eq. (3.10)

$$G_i^1(\underline{x}) = \phi(r)m_i . \quad (4.3)$$

It is readily noted that Eq. (4.2) may be interpreted as the Galerkin vector leading to a solution which corresponds to a concentrated load acting at  $r = 0$  if it is specified that

$$\lim_{n \rightarrow \infty} \int \rho F_i^{(n)}(\xi) dV^{(n)}(\xi) = L_i . \quad (4.4)$$

In the following, the fundamental displacements, displacement gradients, strains, and rotations are given. Throughout the remainder of this section,  $i \neq j \neq k$  and  $i, j, k$  will not be summed.

The strain-displacement relations are given by

$$U_i^1 = \frac{1}{8\pi\mu(1-\nu)r^3} \left[ 4(1-\nu) m_i x_i^2 + (3-4\nu) m_i x_j^2 + (3-4\nu) m_i x_k^2 + m_j x_i x_j + m_k x_i x_k \right] . \quad (4.5)$$

The displacement gradients are

$$U_{i,i}^1 = \frac{1}{8\pi\mu(1-\nu)r^5} \left[ 4(\nu-1) m_i x_i^3 + m_j x_j^3 + m_k x_k^3 - 2m_j x_i^2 x_j - 2m_k x_i^2 x_k + (4\nu-1) m_i x_i x_j^2 + (4\nu-1) m_i x_i x_k^2 + m_j x_j x_k^2 + m_k x_j x_k^2 \right] . \quad (4.6)$$

However, for the off-diagonal terms,

$$U_{i,j}^1 = \frac{1}{8\pi\mu(1-\nu)r^5} \left[ m_j x_i^3 + (4\nu-3) m_i x_j^3 + (4\nu-6) m_i x_i^2 x_j - 2m_j x_i x_j^2 + m_j x_i x_k^2 + (4\nu-3) m_i x_j x_k^2 - 3m_k x_i x_j x_k \right] \quad (4.7)$$

Next, the strains are found to be

$$E_{ii}^1 = \frac{1}{8\pi\mu(1-\nu)r^5} \left[ 4(i-1) m_i x_i^3 + m_j x_j^3 + m_k x_k^3 - 2m_j x_i^2 x_j - 2m_k x_i^2 x_k + (4i-1) m_i (x_i x_j^2 + x_i x_k^2) + m_j x_j x_k^2 + m_k x_j x_k^2 \right] . \quad (4.8)$$

For the off-diagonal terms,

$$E_{ij}^1 = \frac{1}{8\pi\mu(1-\nu)r^5} \left[ (2\nu-1) m_j x_i^3 + (2\nu-1) m_i x_j^3 + (2\nu-4) m_i x_i^2 x_j \right. \\ \left. + (2\nu-4) m_j x_i x_j^2 + (2\nu-1) m_j x_i x_k^2 + (2\nu-1) m_i x_j x_k^2 - 3m_k x_i x_j x_k \right] = E_{ji}^1 . \quad (4.9)$$

Finally, the rotations are

$$\Omega_{ij}^1 = \frac{1}{8\pi\mu(1-\nu)r^5} \left[ 4(1-\nu) m_j x_i^3 + 4(\nu-1) m_i x_j^3 + 4(1-\nu) m_i x_i^2 x_j \right. \\ \left. + 4(1-\nu) m_j x_i x_j^2 + 4(1-\nu) m_j x_i x_k^2 + 4(\nu-1) m_i x_j x_k^2 \right] = -\Omega_{ji}^1 . \quad (4.10)$$

So, the displacements, displacement gradients, strains, and rotations for a state of stress with a first order isolated singularity at the origin have been defined in terms of  $m_i$ ,  $u$ ,  $\nu$ , and the coordinates of the point.

It is now possible to obtain higher order isolated singularities by a similar superposition technique. Flexibility influence coefficients  $A_{ij}(\xi)$  can be used to describe the effects of a force  $Q_j/c$  acting at a po.  $\xi$  on the displacement  $u_j$  at some other point  $x$  (Figure 1). In fact, the displacement is simply

$$U_i^1(x) \Big|_{\xi} = A_{ij}(x, \xi) \frac{Q_j}{c} \quad . \quad (4.11)$$

Under conditions shown in Figure 1, Eq. (4.11) will be identical to Eq. (4.5). This represents the fundamental or first order singular solution. Similarly, the displacement at  $x$  because of a concentrated force  $-Q_j/c$  acting at  $\xi + c$  is

$$U_i^1(x) \Big|_{\xi+c} = -A_{ij}(x, \xi + c) \frac{Q_j}{c} \quad . \quad (4.12)$$

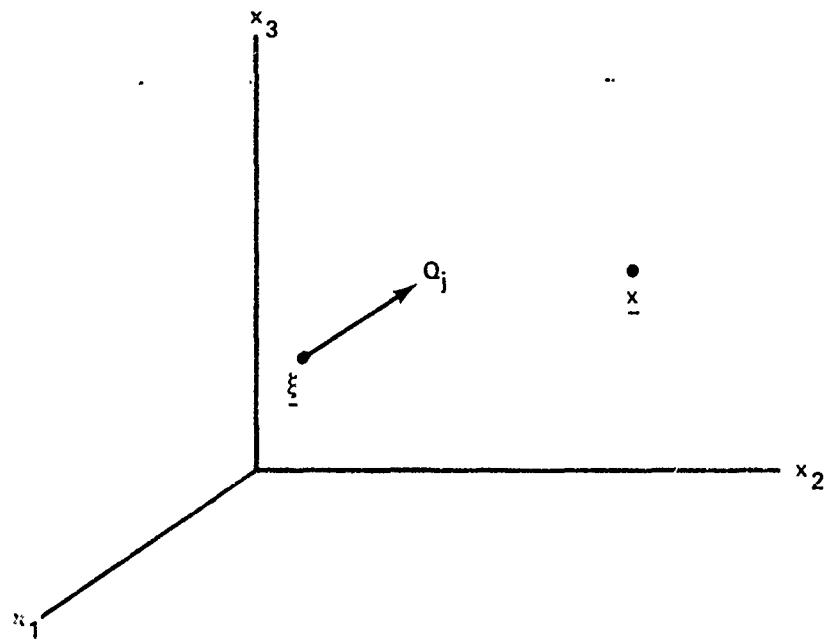


Figure 1. Concentrated Load at  $\xi$ , First Order Singularity

Because the force-displacement relationship is considered linear, the displacements created by  $Q_j/\xi$  at  $\xi$  and  $-Q_i/\xi$  at  $\xi + \xi$  are simply the superposition of Eqs. (4.11) and (4.12) (Figure 2). Taking the limit of the resulting expression yields a second order singular solution

$$U_i^2(\underline{x}) = \lim_{\xi \rightarrow 0} \left[ U_i^1(\underline{x}) \Big|_{\xi} + U_i^1(\underline{x}) \Big|_{\xi + \xi} \right] = \lim_{\xi \rightarrow 0} \frac{A_{ij}(\underline{x}, \xi) - A_{ij}(\underline{x}, \xi + \xi)}{\xi} Q_j \quad (4.13)$$

which is recognized as the first derivative of  $U_i^1(\underline{x})$ .

Specifically, in Figure 3,  $Q$  is parallel to the  $x_1$  axis and  $\xi$  is measured along the  $x_1$  axis. Therefore, let  $U_i^1(\underline{x}) \Big|_{\xi}$  and  $U_i^1(\underline{x}) \Big|_{\xi + \xi}$  be represented by Eq. (4.5) where only  $m_1$  is nonzero. For convenience, the point  $\xi$  is designated at the origin. Because the influence coefficients are dependent upon the vector  $R(\underline{x}, \xi)$ , the displacements at  $\underline{x}$  caused by the forces shown are

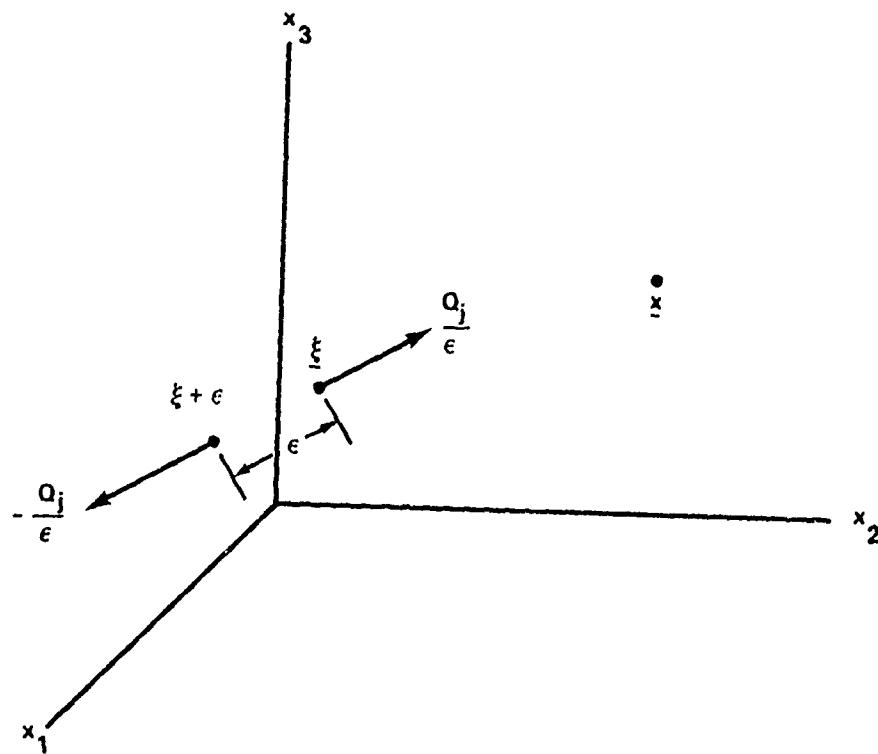


Figure 2. Second Order Singularity

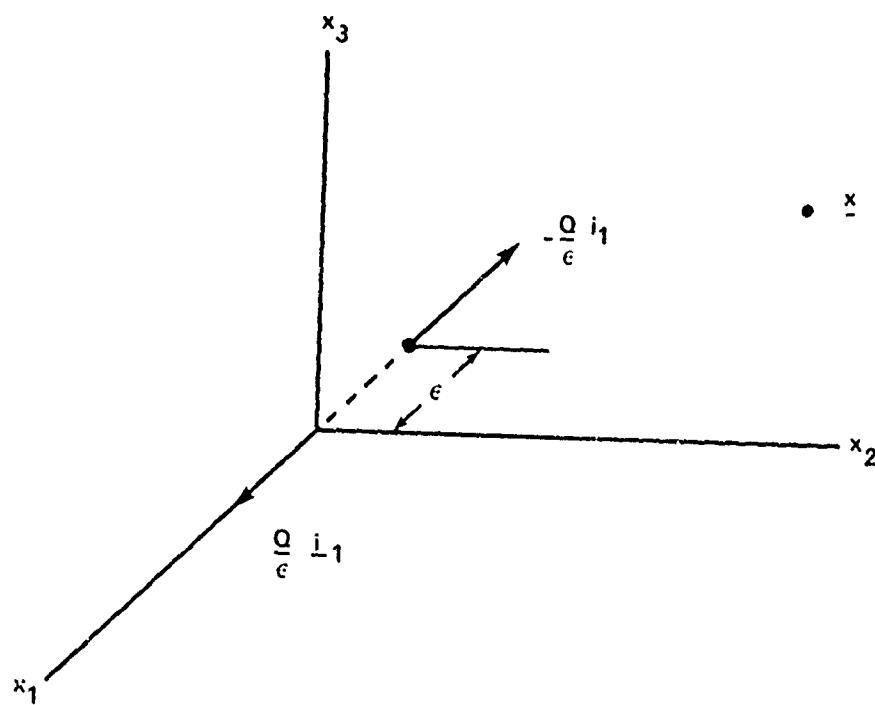


Figure 3. Force Doublet

$$\begin{aligned}
 U_i^2(\underline{x}) &= \lim_{\epsilon \rightarrow 0} \left[ \bar{U}_i^1(\underline{x}) \Big|_{\xi} + \bar{U}_i^1(\underline{x}) \Big|_{\xi+\epsilon} \right] \\
 &= \lim_{\epsilon \rightarrow 0} \frac{A_i(x_1, x_2, x_3) - A_i[(x_1 - \epsilon), x_2, x_3]}{\epsilon} Q = U_{i,1}^1(\underline{x}) \quad . \tag{4.14}
 \end{aligned}$$

This is referred to as a force doublet.\* If the forces are arranged as shown in Figure 4 then the displacement field is given by

$$U_i^2(\underline{x}) = \lim_{\epsilon \rightarrow 0} \frac{A_i(x_1, x_2, x_3) - A_i[(x_1, x_2 - \epsilon), x_3]}{\epsilon} Q = \hat{U}_{i,2}^1(\underline{x}) \quad (4.15)$$

where  $\hat{U}_{i,2}^1(\underline{x})$  is represented by Eq. (4.5) if only  $m_1$  is nonzero and is called a force couplet.

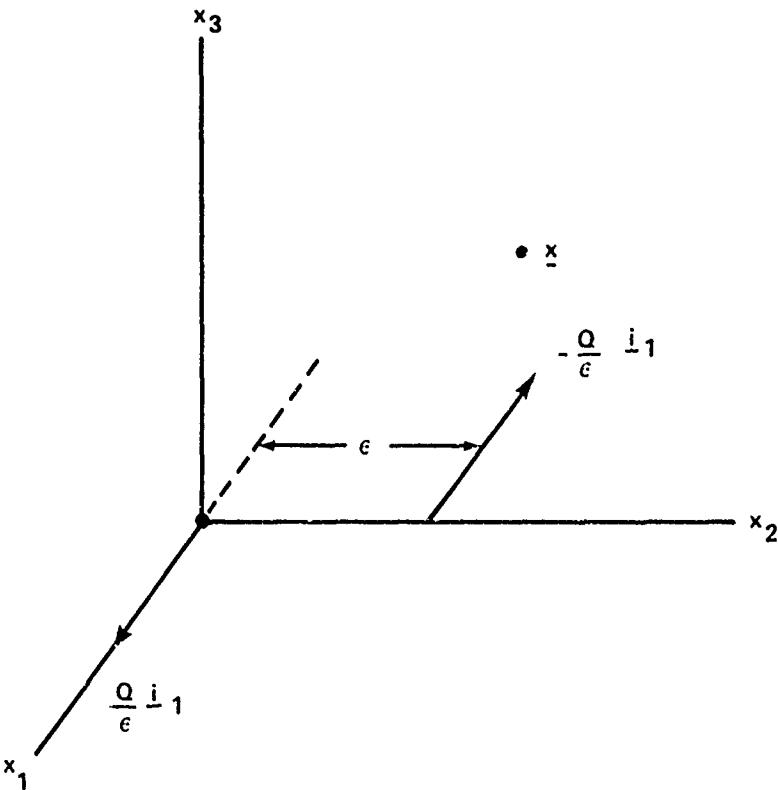


Figure 4. Force Couplets

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\*Three force doublets of equal magnitude acting along the  $x_1$ ,  $x_2$ , and  $x_3$  axes comprise a center of dilatation.

The generalization of this process is intuitively simple. Doublets and couplets of any magnitude may be generated in various directions or may themselves be superposed to form third or higher order singularities. This procedure is, however, artificial and tedious. In Paragraphs 5 and 6, the limit solution technique will be employed to form expressions for all possible second and third order isolated singular solutions.

### 5. Second Order Singularities

It is now observed that the terms  $\Phi(r)$ ,  $\Phi(r)$ ,  $\Phi(r)$  and  $M^{100}$ ,  $M^{010}$ ,  $M^{001}$  can be treated as components of first and second order tensors. Making the following definitions

$$\phi_j(r) = -\frac{\alpha x_j}{r}, \quad m_{ij} = \lim_{n \rightarrow \infty} \int \rho F_i^{(n)}(\xi) \xi_j dV^{(n)}(\xi). \quad (5.1)$$

The Galerkin vector corresponding to second order singularities may be written as

$$G_i^2(x) = \phi_j(r) m_{ij} \quad (5.2)$$

which is identical to the second term of Eq. (3.10).

To examine the physical characteristics of various second order singularities, one may first allow all  $m_{ij}$  except  $m_{11}$  to be zero. In this case, Eq. (5.2) becomes

$$G^2(x) = [\phi_1(r) m_{11}, 0, 0] \quad . \quad (5.3)$$

One may say that the resulting displacement pertains to a force doublet along the  $x_1$  axis.\* Similarly if  $m_{22}$  or  $m_{33}$  are the only nonzero components of  $m_{ij}$ , then the doublets are along the  $x_2$  or  $x_3$  axis, respectively. The significance of the off-diagonal terms is shown by setting all  $m_{ij}$  equal to zero except  $m_{23}$ . Now the Galerkin vector is

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\*If only  $m_{11}$ ,  $m_{22}$ ,  $m_{33}$  are nonzero and  $m_{11} = m_{22} = m_{33}$ , a center of dilatation is produced.

$$\underline{G}^2(\underline{x}) = [0, \phi_3(r) m_{23}, 0] . \quad (5.4)$$

This corresponds to a force couplet about the  $x_1$  axis with the forces in the  $x_2$  direction.

If  $\rho F_i^{(n)}$  are symmetric about all axes then all  $m_{ij}$  are zero and, hence, there are no second order singular terms. If  $\rho F_i^{(n)}$  are symmetric about one or two axes then certain components of the tensor  $m_{ij}$  will be zero. For example, if  $\rho F_i^{(n)}$  is symmetric about the plane perpendicular to the  $x_3$  axis then  $m_{13} = m_{23} = m_{33} = 0$  and the resulting Galerkin vector is

$$\underline{G}^2(\underline{x}) = \left\{ [\phi_1(r)m_{11} + \phi_2(r)m_{12}], [\phi_1(r)m_{21} + \phi_2(r)m_{22}], [\phi_1(r)m_{31} + \phi_2(r)m_{32}] \right\} . \quad (5.5)$$

This indicates doublets on the  $x_1$  and  $x_2$  axes as well as couplets about the  $x_1$ ,  $x_2$ , and  $x_3$ .

If  $\rho F_i^{(n)}$  are symmetric relative to planes along two axes (for example, the  $x_2$  and  $x_3$  axes) then only the  $m_{11}$ ,  $m_{21}$ , and  $m_{31}$  components are nonzero. Hence, the Galerkin vector becomes

$$\underline{G}^2(\underline{x}) = [\phi_1(r)m_{11}, \phi_1(r)m_{21}, \phi_1(r)m_{31}] \quad (5.6)$$

which corresponds to a doublet on the  $x_1$  axis and couplets about the  $x_2$  and  $x_3$  axes.

Every second order tensor may be expressed as the sum of a symmetric and antisymmetric tensor. Therefore,  $m_{ij}$  may be written as

$$m_{ij} = m_{ij}^S + m_{ij}^A \quad (5.7)$$

where

$$m_{ij}^S = \frac{1}{2}(m_{ij} + m_{ji}), \quad m_{ij}^A = \frac{1}{2}(m_{ij} - m_{ji}), \quad (5.8)$$

$$m_{ij}^A = \frac{1}{2}(m_{ij} - m_{ji}), \quad m_{ij}^A = -m_{ji}^A . \quad (5.9)$$

Now, the products of  $\phi_j(r)$  and  $m_{ij}^S$  are given below:

$$G_i^{2S}(x) = \phi_j(r) m_{ij}^S = -\frac{\alpha}{r} x_j m_{ij}^S . \quad (5.10)$$

Similarly, the products of  $\phi_j(r)$  and  $m_{ij}^A$  are

$$G_i^{2A}(x) = \phi_j(r) m_{ij}^A = -\frac{\alpha}{r} x_j m_{ij}^A . \quad (5.11)$$

The displacement fields corresponding to Eqs. (5.10) and (5.11) are

$$\begin{aligned} u_i^{2S} &= \frac{1}{16\pi\mu(1-\nu)r^5} \left\{ \left[ 4(\nu-1) m_{ii}^S + m_{jj}^S + m_{kk}^S \right] x_i^3 - (4\nu-2) m_{ij}^S x_j^3 \right. \\ &\quad + (4\nu-2) m_{ik}^S x_k^3 + 4(\nu-1) m_{ij}^S x_i^2 x_j + 4(\nu-1) m_{ik}^S x_i^2 x_k \\ &\quad + \left[ (4\nu-1) m_{ii}^S - 2m_{jj}^S + m_{kk}^S \right] x_i^2 x_j^2 + \left[ (4\nu-1) m_{ii}^S + m_{jj}^S - 2m_{kk}^S \right] x_i^2 x_k^2 \\ &\quad \left. + (4\nu-2) m_{ik}^S x_j^2 x_k + (4\nu-2) m_{ij}^S x_j^2 x_k - 6m_{jk}^S x_i^2 x_j x_k \right\} \\ &\quad \text{--- no sum, } i \neq j \neq k \end{aligned} \quad (5.12)$$

$$\begin{aligned} u_i^A &= \frac{1}{16\pi\mu(1-\nu)r^5} \left\{ 4(1-\nu) m_{ij}^A x_j^3 + 4(1-\nu) m_{ik}^A x_k^3 + 4(1-\nu) m_{ij}^A x_i^2 x_j \right. \\ &\quad + 4(1-\nu) m_{ik}^A x_i^2 x_k + 4(1-\nu) m_{ik}^A x_j^2 x_k + 4(1-\nu) m_{ij}^A x_j^2 x_k \left. \right\} \\ &\quad \text{--- no sum, } i \neq j \neq k . \end{aligned} \quad (5.13)$$

It is easily shown from Eqs. (4.8) and (5.12) that if  $m_1$  is the only nonzero component in Eq. (4.8) and if  $m_{11}^S = m_1$  and is the only nonzero element in the  $m_{ij}^S$  tensor, then

$$v_1^{2S} = E_{11}^1 . \quad (5.14)$$

This can, of course, be generalized for the second and third components of  $\underline{u}^{2S}$  as well. Thus, under these conditions  $U_i^{2S}$  and  $E_{ii}^1$  correspond to a force doublet in the  $x_i$  direction.

Additionally, if  $m_1 = m_2 = m_3$  in Eq. (4.8) and  $m_{11}^S = m_{22}^S = m_{33}^S$  and are the only nonzero terms in  $m_{ij}^S$ , then

$$U_1^{2S} + U_2^{2S} + U_3^{2S} = E_{11}^1 + E_{22}^1 + E_{33}^1 \quad (5.15)$$

and represent a center of dilatation. Further, from Eqs. (5.10) and (5.13) it can be shown that each component of the antisymmetric part of the second order singularity is identical to the rotation vector about its axis. That is (if the signs of  $m_{ij}^A$  are dictated by the right hand rule),

$$U_i^{2A} = \Omega_{jk}^1, \quad i \neq j \neq k \quad (5.16)$$

and corresponds to a force couplet about the  $x_i$  axis.

## 6. Third Order Singularities

The third term in Eq. (3.10) can also be expanded and specialized for particular cases. The expansion is shown as

$$G_i^3(\underline{x}) = \epsilon_{ijk}(r) m_{ijk} \quad (6.1)$$

where

$$\epsilon_{ij}(r) = \alpha \left[ \frac{\underline{i}\underline{j}}{r} - \frac{\underline{x}_i \underline{x}_j}{r^3} \right], \quad m_{ijk} = \lim_{n \rightarrow \infty} \int \rho F_i^{(n)}(\underline{z}) \epsilon_{jkl} dV^{(n)}(\underline{z}). \quad (6.2)$$

Only one particular form of third order singularity will be presented; however, it is obvious that one can easily express any specific type. For example, the following Galerkin vector corresponds to a singularity at the origin which can be called a double center of dilatation on the  $x_3$  axis if

$$\underline{G}^3(\underline{x}) = [\phi_{13}(r)m_{131}, \phi_{23}(r)m_{232}, \phi_{33}(r)m_{333}] \quad (6.3)$$

where  $m_{131} = m_{232} = m_{333}$ .

### 7. Spherical Biharmonics

The series represented by Eq. (3.10) provides the foundation for a development analogous to that of the class of Newtonian potentials called spherical harmonics. It may be recalled that a Newtonian potential is a solution to Laplace's equation at points external to the body. For a body of density  $\sigma(\xi)$  occupying the region  $V + \Delta V$  in space, the Newtonian potential at  $\underline{x}$  is

$$N(\underline{x}) = \int \frac{\sigma(\underline{\xi})}{R(\underline{x}, \underline{\xi})} dV(\underline{\xi}) \quad (7.1)$$

By expanding  $1/R(\underline{x}, \underline{\xi})$  about  $\underline{x}$  in a Taylor Series,  $N$  may be written as

$$N = \sum_{i,j,k=0}^{\infty} B_{ijk} \left( \frac{1}{r} \right) I_{ijk} \quad (7.2)$$

where

$$B_{ijk} \left( \frac{1}{r} \right) = \frac{(-1)^{i+j+k}}{i!j!k!} D_1^i D_2^j D_3^k \left( \frac{1}{r} \right) \quad (7.3)$$

$$I_{ijk} = \int \left( \frac{1}{r} \right)^{i+j+k} dV(\underline{\xi}) \quad (7.4)$$

It is easily seen that  $B_{ijk}(1/r)$  is homogeneous of degree  $-(i+j+k+1)$ . Further,  $1/r$  is harmonic. Therefore, each  $B_{ijk}(1/r)$  must be harmonic because  $\nabla^2$  operates only on  $x_i$ . Thus, by definition,  $B_{ijk}(1/r)$  are spherical harmonics of degree  $-(i+j+k+1)$ .

A comparison of Eqs. (7.2) and (3.9) enables one to make several observations. First, Eq. (7.2) represents the potential function of an attractive force on  $\underline{x}$  by the mass occupying region  $V$ , while Eq. (3.9) is a potential function of the displacement at a point  $\underline{x}$  resulting from a force or moment acting at the origin. It was shown that the coefficients  $B_{ijk}(1/r)$  are spherical harmonics and by similar reasoning it is noted that  $\Phi_{ijk}(r)$  are biharmonic and homogeneous of degree  $(l-i-j-k)$ , a function which will be called a spherical biharmonic of degree  $(l-i-j-k)$ .

Thus, with the analogy clearly established, an interesting point may be investigated by first noting that the expansion of Eq. (7.1) yields

$$\begin{aligned} N = & \frac{1}{r} \int \sigma \, dV(\underline{\xi}) + \frac{x_1}{r} \int \sigma \xi_1 \, dV(\underline{\xi}) + \frac{x_2}{r} \int \sigma \xi_2 \, dV(\underline{\xi}) \\ & + \frac{x_3}{r} \int \sigma \xi_3 \, dV(\underline{\xi}) + O(\xi_i^2) + O(\xi_i^3) \\ & + O(\xi_i^4) + \dots \quad . \end{aligned} \quad (7.5)$$

All terms except the fundamental singularity may be eliminated by prescribing that

$$\int \sigma \xi_1^i \xi_2^j \xi_3^k \, dV(\underline{\xi}) = 0 \quad \text{for } i \neq 0, j \neq 0, k \neq 0 \quad . \quad (7.6)$$

This step is sufficient to insure that  $N$  is the fundamental singularity, but is it necessary? One suspects that it is not when considering the following: A complete, homogeneous, polynomial of degree  $n$  contains  $(n+1)(n+2)/2$  coefficients, only  $2n+1$  of which are arbitrary [9] (assuming that all coefficients are harmonic). For example, the  $O(\xi_i^2)$  terms are

$$\begin{aligned}
& \frac{1}{2r^5} \left[ \left( 2x_1^2 - x_2^2 - x_3^2 \right) \int \sigma \xi_1^2 dV(\underline{\xi}) + \left( -x_1^2 + 2x_2^2 - x_3^2 \right) \int \sigma \xi_2^2 dV(\underline{\xi}) \right. \\
& + \left( -x_1^2 - x_2^2 + 2x_3^2 \right) \int \sigma \xi_3^2 dV(\underline{\xi}) + 6x_1 x_2 \int \sigma \xi_1 \xi_2 dV(\underline{\xi}) \\
& \left. + 6x_1 x_3 \int \sigma \xi_1 \xi_3 dV(\underline{\xi}) + 6x_2 x_3 \int \sigma \xi_2 \xi_3 dV(\underline{\xi}) \right] . \quad (7.7)
\end{aligned}$$

The second degree complete, homogeneous, harmonic polynomial inside the brackets has six terms.\* The fact that there are dependent coefficients in terms of order greater than 1 suggests that conditions other than Eq. (7.6) will result in the elimination of higher order terms. This lack of uniqueness can be illustrated by allowing

$$\int \sigma \xi_1^2 dV(\underline{\xi}) = \int \sigma \xi_2^2 dV(\underline{\xi}) = \int \sigma \xi_3^2 dV(\underline{\xi}) \quad (7.8)$$

in Eq. (7.7). In this case the first three terms in Eq. (7.5) are zero. The  $O(\xi_i^2)$  terms may be written in powers of  $x_i$ ; that is,

$$\begin{aligned}
& \frac{1}{2r^5} \left[ \left( 2\bar{M}_1 - \bar{M}_2 - \bar{M}_3 \right) x_1^2 + \left( -\bar{M}_1 + 2\bar{M}_2 - \bar{M}_3 \right) x_2^2 + \left( -\bar{M}_1 - \bar{M}_2 + 2\bar{M}_3 \right) x_3^2 \right. \\
& \left. + \bar{M}_{12} x_1 x_2 + \bar{M}_{13} x_1 x_3 + \bar{M}_{23} x_2 x_3 \right] \quad (7.9)
\end{aligned}$$

where

$$\bar{M}_{ij} = \int \sigma \xi_i \xi_j dV(\underline{\xi}), \quad i \neq j \quad (7.10)$$

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\*The bracketted expression may be discussed with generality because if  $B_n$  is a spherical harmonic, then  $B_n' r^{2n+1}$  is also a spherical harmonic [9].

and

$$\bar{M}_i \equiv \int \sigma \xi_i^2 dV(\xi) . \quad (7.11)$$

Now the first three terms of Eq. (7.9) are zero if

$$\{\bar{M}\}[K] \equiv \begin{Bmatrix} \bar{M}_1 \\ \bar{M}_2 \\ \bar{M}_3 \end{Bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} = 0 . \quad (7.12)$$

Thus, a condition other than Eq. (7.6) exists which eliminates the  $O(\xi_i^2)$  terms.

The  $O(\xi_i^3)$  term has 10 coefficients. One can write

$$\begin{aligned} & \frac{1}{2r^7} \left\{ \int \sigma \left( 2\xi_1^3 - 3\xi_1\xi_2^2 - 3\xi_1\xi_3^2 \right) dV(\xi) x_1^3 + \int \sigma \left( -3\xi_1^3 + 12\xi_1\xi_2^2 - 3\xi_1\xi_3^2 \right) dV(\xi) x_1 x_2^2 \right. \\ & + \int \sigma \left( -3\xi_1^3 - 3\xi_1\xi_2^2 + 12\xi_1\xi_3^2 \right) dV(\xi) x_1 x_3^2 + \int \sigma \left( 2\xi_2^3 - 3\xi_2^2 \xi_1 - 3\xi_2^2 \xi_3 \right) dV(\xi) x_2^3 \\ & + \int \sigma \left( -3\xi_2^3 + 12\xi_2^2 \xi_1 - 3\xi_2^2 \xi_3 \right) dV(\xi) x_1^2 x_2 + \int \sigma \left( -3\xi_2^3 - 3\xi_1^2 \xi_2 + 12\xi_2^2 \xi_3 \right) dV(\xi) x_2 x_3^2 \\ & + \int \sigma \left( 2\xi_3^2 - 3\xi_3^2 \xi_1 - 3\xi_3^2 \xi_2 \right) dV(\xi) x_3^3 + \int \sigma \left( -3\xi_3^3 + 3\xi_1^2 \xi_3 - 3\xi_3^2 \xi_2 \right) dV(\xi) x_1^2 x_3 \\ & \left. + \int \sigma \left( -3\xi_3^3 - 3\xi_3^2 \xi_1 + 12\xi_3^2 \xi_2 \right) dV(\xi) x_2^2 x_3 + \int 30 \sigma \xi_1 \xi_2 \xi_3 dV(\xi) x_1 x_2 x_3 \right\} . \quad (7.13) \end{aligned}$$

It may be seen that the entire term vanishes if all integrals are zero, or if

$$\int \frac{\sigma}{3} r_i^3 dV(\underline{r}) = \int \sigma r_j^2 dV(\underline{r}) = \int \sigma r_k^2 dV(\underline{r}) = \alpha_i, \quad i \neq j \neq k$$

$$\int \sigma r_1 r_2 r_3 dV(\underline{r}) = 0 \quad (7.14)$$

and so on for all orders.

To visualize the theoretical possibility of such an occurrence, imagine a sphere of radius  $c$  centered at 0 and a spherical shell with an outer radius  $a$  and an inner radius  $b$  also centered at 0. If  $b < c$  the sphere is encapsulated inside the shell. The potential of attraction of the sphere relative to a point  $\underline{x}$  lying outside the sphere is

$$N_0 = \frac{4\pi\sigma}{r(\underline{x})} \int_0^c [r(\underline{r})]^2 dr(\underline{r}) = \frac{4\pi\sigma}{3r(\underline{x})} c^3 \quad . \quad (7.15)$$

Similarly, the potential of the shell relative to a point  $\underline{x}$  lying inside of the inner wall is

$$N_I = 4\pi\sigma \int_b^a r(\underline{r}) dr(\underline{r}) = 2\pi\sigma (a^2 - b^2) \quad . \quad (7.16)$$

Of course, the potential of a point lying between the sphere and shell is  $N = N_0 + N_I$ .

If one prescribes

$$\cdot^1 = - \frac{2c^3}{3(a^2 - b^2)} \quad . \quad (7.17)$$

then  $N$  is zero without requiring that all integrals vanish.

The preceding investigation was conducted with a harmonic function  $\psi$ . Its results, however, can be applied to a biharmonic function  $g$  by noting that an arbitrary harmonic function  $\beta$  can be defined so that

$$\nabla^2 g = \beta . \quad (7.18)$$

Thus, in the expansion of the biharmonic vector  $g$  there will be dependencies among coefficients beginning with the  $O(\cdot_i^4)$  term, because this corresponds to the  $O(\cdot_i^2)$  term in the expansion of  $\beta$ .

### 8. Betti's Reciprocal Work Theorem

In Paragraph 2, Galerkin's vector  $g(x_i)$  was used in discussing the development of a particular solution of the Cauchy's equations. In general, these functions are analogous to potential functions and, as stated previously, several techniques and theorems of potential theory are applicable in the treatment of the equations of elasticity. A lucid explanation of the application of potential theory to the solution of elliptical equations is found in Courant and Hilbert [3]. One of the most important aspects of potential theory is the theorem and identities of Green. Betti [1], in 1872, adapted Green's identities to elasticity and thus formulated the reciprocal work theorem. In this section the reciprocal work theorem and a particular application will be discussed.

Consider two stress states  $S(x)$  and  $S'(x)$  which are, for the moment, assumed to be regular throughout an elastic body  $V$ . If the unprimed state has a zero body force, the equations of equilibrium for  $S(x)$  and  $S'(x)$  are

$$h_{ijkl} u_{k,lj} = 0 \quad (8.1)$$

$$h_{ijkl} u'_{k,lj} + \gamma'_i = 0 . \quad (8.2)$$

Multiplying Eq. (8.1) by  $u'_i$  and Eq. (8.2) by  $u_i$ , respectively, and subtracting the resulting expression yields a relationship between the primed and unprimed states. Integrating this relationship over  $V$  results in the following:

$$\int h_{ijkl} [u_{k,lj} u'_i - u'_{k,lj} u_i] dV + \int \gamma'_i u_i dV = 0 . \quad (8.3)$$

Finally, employing Green's theorem and noting that

$$\int H_{ijkl} u_{k,l} \eta_j u_i' dV = \int T_i u_i' dV , \quad (8.4)$$

one may write the reciprocal work theorem as

$$\int [T_i u_i' - T_i' u_i] dV + \int \rho F_i' u_i dV = 0 . \quad (8.5)$$

Imagine now that lying within the region  $V$  is the sequence of regions  $\{V^{(n)}\}$  having the properties described in Paragraph 3. Suppose further that the primed stress state corresponds to  $S^{(n)}(\underline{x})$ , where  $\rho F_i^{(n)}(\underline{x})$  is zero for  $\underline{x}$  not in  $V^{(n)}$ . For the region  $V^{(n)}$ , Eq. (8.5) becomes

$$\int [T_i u_i^{(n)} - T_i^{(n)} u_i] dV^{(n)} - \int \rho F_i^{(n)} u_i dV^{(n)} = 0 . \quad (8.6)$$

Similarly, applying Eq. (8.5) over the region  $V - V^{(n)}$  yields

$$\begin{aligned} & - \int [T_i u_i^{(n)} - T_i^{(n)} u_i] dV + \int [T_i u_i^{(n)} - T_i^{(n)} u_i] dV^{(n)} \\ & + \int \rho F_i^{(n)} u_i d(V - V^{(n)}) = 0 . \end{aligned} \quad (8.7)$$

Note that the tractions over  $V^{(n)}$  as well as  $V$  must be considered. The terms  $T_i^{(n)}$  and  $T_i$  involve an outer normal in Eq. (8.6) and an inner normal in the second integral of Eq. (8.7); therefore, the signs are opposite. The addition of these two equations results in

$$\int [T_i u_i^{(n)} - T_i^{(n)} u_i] dV - \int \rho F_i^{(n)} u_i dV^{(n)} = 0 . \quad (8.8)$$

Expanding  $u_i$  about the origin in a Taylor's series and taking the limit as  $n$  approaches infinity yields

$$0 = \lim_{n \rightarrow \infty} \int \left[ T_i u_i^{(n)} - T_i^{(n)} u_i \right] dV + u_i(0) \lim_{n \rightarrow \infty} \int \rho F_i^{(n)} dV^{(n)}$$

$$+ u_{i,j}(0) \lim_{n \rightarrow \infty} \int \rho F_i^{(n)} \xi_j dV^{(n)} + u_{i,jk}(0) \lim_{n \rightarrow \infty} \int \rho F_i^{(n)} \xi_j \xi_k dV^{(n)}$$

$$+ \dots \quad (8.9)$$

or

$$0 = \int \left[ T_i U_i - \tau_i u_i \right] dV + u_i(0)m_i + u_{i,j}(0)m_{ij} + u_{i,jk}(0)m_{ijk} + \dots \quad (8.10)$$

where

$$\tau_i = \lim_{n \rightarrow \infty} T_i^{(n)}$$

And so, by choosing the appropriate order singularity (by choosing values of  $m_i$ ,  $m_{ij}$ , etc.), the displacements or any order derivatives of the displacement field of a regular stress state acting throughout can be reproduced if the displacements and tractions are known on  $\partial V$ . Note that by choosing a particular isolated singular solution  $U_i$ , one correspondingly chooses the form of  $m_i$ ,  $m_{ij}$ ,  $m_{ijk}$ , etc.

## 9. Discussion

As stated previously, the object of this report is to offer a physical interpretation of various first and higher order isolated singular solutions of Cauchy's equations using Galerkin's vector. These singularities are constructed by means of a limit solution proposed by Sternberg and Eubanks [8]. Some forms of first, second, and third order isolated singularities are investigated and will be followed by a discussion of uniqueness and an application of Betti's reciprocal theorem. In the following paragraphs, various salient points will be reviewed.

In Paragraph 2 Cauchy's equations are rewritten in a form more easily solved. This form is a nonhomogeneous biharmonic equation in terms of an auxiliary vector called Galerkin's vector. The solution is derived by employing a technique analogous to one used in potential theory.

Next, the limit solution is discussed and a series is developed which contains all first order isolated singular solutions to Cauchy's equations in the first set of terms, all second order singularities in the second set of terms, etc. An investigation of the first order singularity leads to its employment in a superposition method to construct various second order singularities.

Paragraph 5 contains an examination of the second term of Eq. (4.10) and hence, the development of the second order singularities.

It is shown in Paragraph 4 that the first term of Eq. (4.10) may be thought of as corresponding to a concentrated load at the origin. An examination of the second term in Eq. (4.10) revealed the following facts:

- a) Each of the diagonal components  $m_{ii}$  (no sum) produces a force doublet along axis  $x_i$
- b) Each of the off-diagonal components  $m_{ij}$  ( $i \neq j \neq k$ ) produces a force couplet about the  $x_k$  axis
- c) Symmetry of  $\rho F_i^{(n)}$  about one or two axes insures that certain components of  $m_{ij}$  vanish.

By dividing  $m_{ij}$  into its symmetric and antisymmetric parts  $m_{ij}^S$  and  $m_{ij}^A$ , it is found that the solution  $U_i^{2S}$  derived from  $m_{ij}^S$  is identical to the strain  $E_{ii}^1$  (no sum) corresponding to the first order singularity and represents a force doublet in the  $x_i$  direction. Further, it is noted that  $U_1^{2S} + U_2^{2S} + U_3^{2S} = E_{11}^1 + E_{22}^1 + E_{33}^1$  and is a center of dilatation.

Next, it is shown that the antisymmetrical part of the second order singularity produces  $U_i^A$  which is identical to the rotation about the  $x_i$  axis  $E_{jk}^1$  corresponding to the fundamental singularity and represents a force couplet about the  $x_i$  axis. Finally, the Galerkin vector producing a third order singular solution is expressed as  $\zeta_{kj}(r)m_{ijk}$  and  $G_i^3(x)$  corresponding to a double center of dilatation is presented.

Next, an analogy is made between the series representing the Galerkin vector pertaining to all isolated singularities and the spherical

harmonics series of potential theory. It is found that the  $G_i(\underline{x})$  represented by  $\int_{ijk}^1(r)M_{ijk}$  can be defined as a spherical biharmonic. Further, it is demonstrated that, because there are dependencies among coefficients of  $G_i(\underline{x})$  beginning with the  $O(\frac{1}{r^4})$  term, there are an infinite number of combinations which will yield any given singular solution.

Finally, furthering the analogy with potential theory, Betti's reciprocal theorem is presented. Two stress states are chosen, one with a zero body force field and one with an isolated singularity at the origin. Applying the reciprocal work theorem, an expression is presented which enables one to reproduce any order derivatives of the displacement field of a regular stress state acting throughout  $V$  if the displacements and tractions are known on  $\partial V$  by simply choosing the appropriate isolated singular solution.

A physical interpretation of the most important isolated singularities (i.e., concentrated force, force doublet, force couplet, center of dilatation, and double center of dilatation) has been given for isotropy. The same approach could undoubtedly be used in the case of anisotropy. As stated previously, proof that the derivatives of the limit value of the auxiliary function must be identical to the limit of the derivatives of the auxiliary function must be given.

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